Numerical Methods Laboratory

Modelling the Climate System
ARC Centre of Excellence for Climate System Science
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Introduction

This laboratory presents an introduction to numerical techniques used in climate and weather models. We will investigate a numerical approach known as finite differences. To illustrate the key points of numerical methods and climate modeling we will focus on the linear advection equation in this laboratory.

Linear Advection

We will study the one dimensional linear advection equation which is given by

$$\frac{\partial u}{\partial t} + \sigma \frac{\partial u}{\partial x} = 0,$$

where $u = u(x, t)$ and σ is a constant. The quantity u could be water vapour, cloud or a chemical species and the equation tells us how u varies in time. This equation has an exact solution $u(x, t) = u_0(x - \sigma t)$ where $u_0(x) = u(x, 0)$ is the initial "pulse" shape before advection begins. This means the pulse simply moves forwards with velocity σ but does not change shape in any way. We will use this property to understand our numerical solutions.

Finite Differencing

To explore numerical methods in climate models we will investigate a technique known as finite differences. This involves computing derivatives by using model grid points separated by Δt and Δx. One finite differencing technique is the so-called "forward in time-centered in space" approach. This takes known values from the previous period at time $t - \Delta t$ and estimates the value at the current period t. The formula for centered differences applied to the linear advection equation is:

$$u_i^n = u_i^{n-1} - \frac{\alpha}{2}\left(u_{i+1}^{n-1} - u_{i-1}^{n-1}\right) + \mathcal{O}(\Delta t, \Delta x^2),$$

where $\alpha = \sigma \Delta t / \Delta x$, with Δt and Δx being the grid intervals in time and position respectively. The index notation is defined by $u_i^n = u(t_n, x_i)$, $u_i^{n-1} = u(t_n - \Delta t, x_i)$ and $u_{i\pm1}^{n-1} = u(t_n - \Delta t, x_i \pm \Delta x)$. The term $\mathcal{O}(\Delta t, \Delta x^2)$ tells us the accuracy of this differencing scheme relative to the grid intervals, with it being 1st order in time and 2nd order in space.

You will find an R implementation (`centered_diff_solution.r`) of this differencing technique on the winter school website. We will use this R program to understand how numerical methods can influence the solution of the linear advection equation. Additionally, in the notes distributed with the laboratory, we provide an analytical analysis of the finite differencing methods considered here. This handout is designed to assist you in understanding the output from the provided finite differencing code.

Part 1: Linear Advection and Centered Differences

To start this laboratory, you will learn how to run the supplied R code and become familiar with linear advection and centered differences.

a) Set up the R code to run the linear advection example with centered differences. You should not need to change any parameters in the R code at this stage. The code is relatively straightforward and is split into the sections: (A) difference parameters, (B) initialization, (C) initial condition and (D) numerical solution. To run the code, open `centered_diff_solution.r`, then use Crtl+A and Crtl+R. A plot of the time evolution of the solution should display.

b) An exact solution to the linear advection equation moves the original pulse forward in the x direction, without any change in shape. Is this occurring for the centered differences numerical solution? If not, what changes are taking place?

c) In the notes distributed with this lab, we show that *centered differences is in fact dispersive*. How does this relate to your observations from b)? What does this say about using centered differences for linear advection over long time periods?

d) Because of the dispersive nature of this scheme, the phase speed of the Fourier modes composing the pulse depend on wave number. Do you notice any modes traveling backwards? Why would this occur? Hint: Consider group velocity.

e) In atmospheric models, water vapour routinely undergoes advection. Consider applying centered differences to this situation. Based on your observations in b) what major problem would occur for water vapor advection using centered differences?

f) Let us now investigate how the solution depends on the differencing parameters: Δx, Δt and σ. Does the numerical solution match the exact solution behaviour for any choice of parameters?

g) When changing the parameters in f) did you notice the solution becoming unstable? The so-called Courant-Friedrichs-Lewy (CFL) criteria tells us that a necessary condition for numerical convergence is $\alpha = \sigma\Delta t/\Delta x < C_{\max}$, where $C_{\max}$ is a constant. What is a representative value of $C_{\max}$ for this application of centered differences? Can you take small or large time steps relative to Δx?

Part 2: Forward and Backwards Differences

In this section, you will modify the differencing method in the supplied R code. We will work with both forward and backwards differencing so make sure you rename the implementations to different file names.

Both forwards and backwards differencing are similar to the centered differencing approach. The expression for forwards differencing is given by

$$u_i^n = u_i^{n-1} - \alpha \left(u_{i+1}^{n-1} - u_i^{n-1} \right) + \mathcal{O}(\Delta t, \Delta x),$$

and for backwards differencing

$$u_i^n = u_i^{n-1} - \alpha \left(u_i^{n-1} - u_{i-1}^{n-1} \right) + \mathcal{O}(\Delta t, \Delta x),$$

where the changes from the centered difference formula are the *different spatial indices and numerical coefficient.* Both of these schemes are accurate to 1^{st} order in space and 1^{st} order in time.

a) Change the R code to implement forward and backwards differencing for the linear advection equation. You will need to change the differences formula and the boundary conditions.

b) Explore forward and backwards differencing using similar approaches as Part 1. Pay particular attention to stability, dispersion and mode propagation. Based on your numerical solutions, compare and contrast the three differencing schemes examined so far. Which method is the best and why?

c) If you were developing a climate model, would your best method be a good choice for the advection scheme? Hint: Consider the other physical properties the scheme should preserve and what might cause it to fail in real world situations.

Optional: The Lax-Wendroff Method

In this optional section we investigate a more advanced finite differencing scheme known as the Lax-Wendroff method. This scheme is quite useful because its errors are 2^{nd} order in both space and time but is efficient and relatively easy to implement. The Lax-Wendroff scheme is derived from a Taylor series expansion to second-order. The scheme takes the form:

$$u_i^n = u_i^{n-1} - \frac{\alpha}{2} \left(u_{i+1}^{n-1} - u_{i-1}^{n-1} \right) + \frac{\alpha^2}{2} \left(u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1} \right) + \mathcal{O}(\Delta t^2, \Delta x^2),$$

where the notation is the same as that used previously. The scheme can be thought of as the centered difference scheme modified at higher order by a derivative term from the Taylor expansion. Climate models typically use between 2^{nd} and 7^{th} order schemes for advection, meaning the Lax-Wendroff method represents one of the basic alternatives for modeling advection in a climate model.

a) Using the supplied R program, code the Lax-Wendroff method to solve the linear advection equation.

b) Now explore the Lax-Wendroff method in a similar manner to the previous sections of the laboratory. Note your observations.

c) Did you find any improvements in the solution using the Lax-Wendroff method? If so, what are they and how are they sensitive to the differencing parameters?

d) Estimate the CFL parameter $C_{\max}$ for this scheme. How does this compare to centered and backwards differences for the same Δx?