Linear Advection Equation

Numerical Methods Laboratory
Modelling the Climate System
ARC Centre of Excellence for Climate System Science
2nd Annual Winter School

18 June 2013

1 Analytical Solution

The linear advection equation is given by

$$\frac{\partial u}{\partial t} + \sigma \frac{\partial u}{\partial x} = 0, \tag{1}$$

where $u = u(x,t)$ and $\sigma =$ constant. To solve the equation we choose cyclic boundary conditions:

$$u(0,t) = u(L,t)$$

and the initial condition:

$$\begin{aligned} u(x,0) &= f(x) \\ &= Ae^{ikx} \qquad \text{for } \ 0 \leq x \leq L, \quad f(x+L) = f(x). \end{aligned}$$

This means the initial condition is just a plane wave. Using separation of variables, we have

$$u(x,t) = X(x)T(t).$$

meaning Eqn. (1) becomes

$$X(x)\frac{\partial T}{\partial t} + \sigma T(t)\frac{\partial X}{\partial x} = 0,$$

and dividing by X(x)T(t)

$$\frac{1}{T(t)}\frac{dT(t)}{dt} = -\sigma\frac{1}{X(x)}\frac{dX(x)}{dt}.$$

This can only have a solution if both sides are constant, as the left hand side (LHS) is a function of t only and the right hand side (RHS) is a function of x only. Let

$$\frac{1}{X}\frac{dX}{dx} = \lambda$$

so that

$$\frac{1}{T}\frac{dT}{dt} = -\sigma\lambda.$$

Therefore,

$$\frac{dX}{dx} = \lambda X \quad \text{and} \quad \frac{dT}{dt} = -\sigma\lambda T.$$

Which have solutions:

$$X(x) = X_0 e^{\lambda x} \quad \text{and} \quad T(t) = T_0 e^{-\sigma\lambda t}$$

so

$$u(x,t) = X(x)T(t) = X_0 T_0 e^{\lambda x - \sigma\lambda t} = u_0 e^{\lambda(x-\sigma t)}.$$

The initial condition gives

$$u(x,0) = u_0 e^{\lambda(x-0)} = Ae^{ikx}$$

so

$$A = u_0 \quad \text{and} \quad \lambda = ik,$$

$$\Rightarrow u(x,t) = u_0 e^{ik(x-\sigma t)}$$

This solution is that of a plane wave that propagates forward in the x direction with velocity σ. This is why the equation is called the linear advection equation – the initial condition is transported forward by the bulk flow with velocity σ.

2 Numerical Solution Using Centered Differences

We now analyze how the centered differences (in space) method changes the solution to the advection equation. Write the linear advection equation from (1) as

$$\frac{\partial u_j}{\partial t} = -\sigma \frac{u_{j+1} - u_{j-1}}{2\Delta x}$$

where

$$u_j = U(t)e^{ijk\Delta x}.$$

Here we have introduced a spatial discretization into the equation. Then

$$\begin{aligned} \tfrac{dU(t)}{dt} e^{ijk\Delta x} &= -\sigma \left\{ \tfrac{U(t)e^{ik(j+1)\Delta x} - U(t)e^{ik(j-1)\Delta x}}{2\Delta x} \right\} \\ &= -\sigma U(t) e^{ijk\Delta x} \left\{ \tfrac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} \right\} \end{aligned}$$

$$\Rightarrow \frac{dU(t)}{dt} = -\sigma U(t) \left\{ \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} \right\}.$$

Euler's formula tells us

$$e^{\pm i\zeta} = \cos\zeta \pm i \sin\zeta,$$

so

$$\frac{dU(t)}{dt} = -\sigma U(t) \frac{2i \sin k\Delta x}{2\Delta x} = -ik\sigma U(t) \frac{\sin k\Delta x}{k\Delta x}$$

$$\Rightarrow \frac{1}{U(t)} \frac{dU(t)}{dt} = \frac{d[\ln(U(t))]}{dt} = -ik\sigma \frac{\sin k\Delta x}{k\Delta x}.$$

Now define

$$\sigma^* = \sigma \frac{\sin k\Delta x}{k\Delta x},$$

and this gives

$$\begin{aligned} \textstyle\int d[\ln(U(t))] &= -\textstyle\int ik\sigma^* dt, \\ \rightarrow \ln(U(t)) - \ln(u_0) &= -ik\sigma^*(t - t_0). \end{aligned}$$

Therefore

$$\ln\left(\frac{U(t)}{u_0}\right) = -ik\sigma^* t, \qquad \text{since } \ u(0) = u_0.$$

$$\Rightarrow U(t) = u_0 e^{-ik\sigma^* t}$$

Substituting back into u_j

$$u_j = u_0 e^{-ik\sigma^* t} e^{ijk\Delta x} = u_0 e^{ik(j\Delta x - \sigma^* t)}.$$

But from the analytical solution we have

$$u(j\Delta x, t) = u_0 e^{ik(j\Delta x - \sigma t)}$$

and since $\sigma \neq \sigma^*$ there is a *discretization error*. Since $\sigma^* = f(k)$, the error is a function of the wavenumber k. This means each wavenumber will behave differently and may not propagate at the same speed σ as in the analytical solution. This is called *numerical dispersion*. The result derived here implies that centered differences is a dispersive numerical scheme.

Let's have a closer look at the "properties" of σ^*. The smallest wavelength that can be resolved in a Δx grid is $\lambda_s = 2\Delta x$ (Figure 1). Since $k = 2\pi/\lambda$,

$$\Rightarrow k_s = 2\pi/2\Delta x = \pi/\Delta x,$$

and

$$\sigma_s^* = \sigma\frac{\sin(k_s\Delta x)}{k_s\Delta x} = \sigma\frac{\sin\pi}{\pi} = 0.$$

No matter what the true phase speed, the shortest waves will have a numerical phase speed of 0. $\Rightarrow$ This implies large error are present for the shortest resolvable waves!

What about longer waves? Say $\lambda = 4\Delta x$,

$$\lambda = 4\Delta x \quad \Rightarrow k = \frac{\pi}{2\Delta x}$$

$$\Rightarrow \sigma^* = \sigma\frac{\sin(\pi/2)}{\pi/2} = 0.6366\sigma.$$

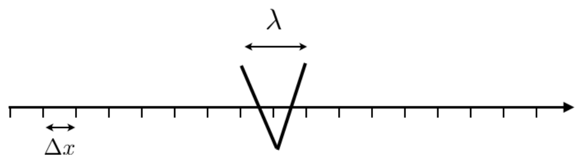

Figure 1: Representation of the minimum resolvable wavelength (λ, the distance from peak to peak) on a numerical grid.

Or say $\lambda = 8\Delta x$,

$$\lambda = 8\Delta x \quad \Rightarrow k = \frac{\pi}{4\Delta x}$$

$$\Rightarrow \sigma^* = \sigma \frac{\sin(\pi/4)}{\pi/4} = 0.9\sigma.$$

This means the longer the wave, the smaller the error in propagation speed.

3 More on Phase Speed

General Case

Let

$$\lambda = n\Delta x$$

$$\sigma^* = \sigma \frac{\sin(2\pi/n)}{2\pi/n}$$

as $n \to \infty$

$$\sigma^*_{n\to\infty} = \lim_{n\to\infty} \sigma \frac{\sin(2\pi/n)}{2\pi/n}.$$

Use L'Hôpital's rule:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}.$$

So

$$\sigma^*_{n\to\infty} = \sigma \frac{\frac{-2\pi}{n^2}\cos(2\pi/n)|_{n=\infty}}{\frac{-2\pi}{n^2}}$$

$$\Rightarrow \sigma^*_{n\to\infty} = \sigma.$$

So the larger n is (ie. the wavelength) the smaller the error!

Group Velocity

In a wave packet (an overlay of many different waves with varying wave numbers) energy is transported with the group velocity

$$\sigma_g = \frac{\partial \omega}{\partial k}$$

where ω = frequency.
The analytical solution is

$$u = u_0 e^{ik(x-\sigma t)} = u_0 e^{i(kx-\omega t)}$$

where

$$\omega = k\sigma.$$

$$\Rightarrow \sigma_g = \frac{\partial}{\partial k}(k\sigma) = \sigma.$$

This tells us that the group velocity equals the phase speed, so the wave is non-dispersive!

Numerical Solution

Lets apply group velocity to the numerical solution:

$$\omega = k\sigma^* = k\sigma\frac{\sin(k\Delta x)}{k\Delta x} = \sigma\frac{\sin(k\Delta x)}{\Delta x}$$

so

$$\sigma^*_g = \frac{\partial \omega}{\partial k} = \sigma\cos(k\Delta x)$$

Example: $\lambda = 2\Delta x \Rightarrow k = \pi/\Delta x$

$$\Rightarrow \sigma_g^* = \sigma \cos \pi = -\sigma.$$

Therefore, the numerical group velocity is not equal to the phase velocity as in the analytical solution. In fact it has the opposite sign!

4 Discretization Error of Backwards Differencing

We now analyse the backward in space differencing scheme. We start by writing the advection equation from (1) as

$$\frac{\partial u}{\partial t} = -\sigma \frac{\partial u}{\partial x} \quad \rightarrow \quad \frac{\partial u_j}{\partial t} = -\sigma \frac{u_j - u_{j-1}}{\Delta x}.$$

Let $u_j = U(t)e^{ijk\Delta x}$

$$\begin{aligned}
\Rightarrow \tfrac{dU(t)}{dt} e^{ijk\Delta x} &= -\sigma \left(\tfrac{U(t)e^{ijk\Delta x} - U(t)e^{i(j-1)k\Delta x}}{\Delta x} \right) \\
\Rightarrow \tfrac{1}{U(t)} \tfrac{dU(t)}{dt} &= -\sigma \left(\tfrac{1 - e^{-ik\Delta x}}{\Delta x} \right) \\
&= -\sigma \left(\tfrac{1 - (\cos k\Delta x - i \sin k\Delta x)}{\Delta x} \right) \\
&= -ik\sigma \left(\tfrac{1 - (\cos k\Delta x - i \sin k\Delta x)}{ik\Delta x} \right) \\
&= -ik\sigma^*
\end{aligned}$$

where

$$\sigma^* = \sigma \left(\frac{1 - \cos k\Delta x + i \sin k\Delta x}{ik\Delta x} \right).$$

Therefore

$$U(t) = u_0 e^{-ik\sigma^* t}$$

and

$$u_j = U(t)e^{ijk\Delta x} = u_0 e^{ik(j\Delta x - \sigma^* t)}$$

Analytic Solution: $u = u_0 e^{ik(x-\sigma t)}$

at $x = j\Delta x \quad \rightarrow u = u_0 e^{ik(j\Delta x - \sigma t)}$.

$\Rightarrow$ Discretization error since $\sigma \neq \sigma^*$.

Investigate σ^* a bit more:

$$
\begin{aligned}
u_j &= u_0 e^{ik\left[j\Delta x - \sigma t\left(\frac{1-\cos k\Delta x}{ik\Delta x}\right) - \sigma t\left(\frac{i \sin k\Delta x}{ik\Delta x}\right)\right]} \\
&= u_0 e^{-\sigma t\left(\frac{1-\cos k\Delta x}{\Delta x}\right)} e^{ik\left[j\Delta x - \sigma t\left(\frac{\sin k\Delta x}{k\Delta x}\right)\right]}.
\end{aligned}
$$

Term containing $\sin k\Delta x / k\Delta x \quad \rightarrow \quad$ numerical dispersion.
Term containing $(1 - \cos k\Delta x)/\Delta x \quad \rightarrow \quad$ damping of solution for $\sigma > 0$, or growth and hence instability for $\sigma < 0$.