1 Analytical Solution

The linear advection equation is given by

$$\frac{\partial u}{\partial t} + \sigma \frac{\partial u}{\partial x} = 0,$$

where $u = u(x, t)$ and $\sigma = \text{constant}$. To solve the equation we choose cyclic boundary conditions:

$$u(0, t) = u(L, t)$$

and the initial condition:

$$u(x, 0) = f(x) = Ae^{ikx} \quad \text{for} \ 0 \leq x \leq L, \ f(x + L) = f(x).$$

This means the initial condition is just a plane wave. Using separation of variables, we have

$$u(x, t) = X(x)T(t),$$

meaning Eqn. (1) becomes

$$X(x)\frac{\partial T}{\partial t} + \sigma T(t)\frac{\partial X}{\partial x} = 0,$$
and dividing by $X(x)T(t)$

$$\frac{1}{T(t)} \frac{dT(t)}{dt} = -\sigma \frac{1}{X(x)} \frac{dX(x)}{dt}. $$

This can only have a solution if both sides are constant, as the left hand side (LHS) is a function of $t$ only and the right hand side (RHS) is a function of $x$ only. Let

$$\frac{1}{X} \frac{dX}{dx} = \lambda$$

so that

$$\frac{1}{T} \frac{dT}{dt} = -\sigma \lambda.$$ 

Therefore,

$$\frac{dX}{dx} = \lambda X \quad \text{and} \quad \frac{dT}{dt} = -\sigma \lambda T.$$ 

Which have solutions:

$$X(x) = X_0 e^{\lambda x} \quad \text{and} \quad T(t) = T_0 e^{-\sigma \lambda t}$$

so

$$u(x,t) = X(x)T(t) = X_0 T_0 e^{\lambda x - \sigma \lambda t} = u_0 e^{\lambda(x-\sigma t)}.$$

The initial condition gives

$$u(x,0) = u_0 e^{\lambda (x-0)} = A e^{ikx}$$

so

$$A = u_0 \quad \text{and} \quad \lambda = ik,$$

$$\Rightarrow u(x,t) = u_0 e^{ik(x-\sigma t)}$$

This solution is that of a plane wave that propagates forward in the $x$ direction with velocity $\sigma$. This is why the equation is called the linear advection equation – the initial condition is transported forward by the bulk flow with velocity $\sigma$. 


2 Numerical Solution Using Centered Differences

We now analyze how the centered differences (in space) method changes the solution to the advection equation. Write the linear advection equation from (1) as

\[
\frac{\partial u_j}{\partial t} = -\sigma \frac{u_{j+1} - u_{j-1}}{2\Delta x}
\]

where

\[ u_j = U(t)e^{ik\Delta x}. \]

Here we have introduced a spatial discretization into the equation. Then

\[
\frac{dU(t)}{dt} e^{ik\Delta x} = -\sigma \left\{ \frac{U(t)e^{ik(j+1)\Delta x} - U(t)e^{ik(j-1)\Delta x}}{2\Delta x} \right\}
\]

\[
= -\sigma U(t)e^{ik\Delta x} \left\{ \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} \right\}
\]

\[
\Rightarrow \frac{dU(t)}{dt} = -\sigma U(t) \left\{ \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} \right\}.
\]

Euler’s formula tells us

\[ e^{\pm i\zeta} = \cos \zeta \pm i \sin \zeta, \]

so

\[
\frac{dU(t)}{dt} = -\sigma U(t) \frac{2i \sin k\Delta x}{2\Delta x} = -ik \sigma U(t) \sin k\Delta x \frac{1}{k\Delta x}
\]

\[
\Rightarrow \frac{1}{U(t)} \frac{dU(t)}{dt} = \frac{d[\ln(U(t))]}{dt} = -ik \sigma \sin k\Delta x \frac{1}{k\Delta x}.
\]

Now define

\[ \sigma^* = \sigma \frac{\sin k\Delta x}{k\Delta x}, \]

and this gives

\[
\int d[\ln(U(t))] = -\int ik \sigma^* dt,
\]

\[
\ln(U(t)) - \ln(u_0) = -ik \sigma^*(t - t_0).
\]
Therefore
\[ \ln \left( \frac{U(t)}{u_0} \right) = -ik\sigma^* t, \quad \text{since } u(0) = u_0. \]

\[ \Rightarrow U(t) = u_0 e^{-ik\sigma^* t} \]

Substituting back into \( u_j \)
\[ u_j = u_0 e^{-ik\sigma^* t} e^{ijk\Delta x} = u_0 e^{ik(j\Delta x - \sigma^* t)}. \]

But from the analytical solution we have
\[ u(j\Delta x, t) = u_0 e^{ik(j\Delta x - \sigma t)} \]
and since \( \sigma \neq \sigma^* \) there is a discretization error. Since \( \sigma^* = f(k) \), the error is a function of the wavenumber \( k \). This means each wavenumber will behave differently and may not propagate at the same speed \( \sigma \) as in the analytical solution. This is called numerical dispersion. The result derived here implies that centered differences is a dispersive numerical scheme.

Let’s have a closer look at the “properties” of \( \sigma^* \). The smallest wavelength that can be resolved in a \( \Delta x \) grid is \( \lambda_s = 2\Delta x \) (Figure 1). Since \( k = 2\pi/\lambda \),
\[ \Rightarrow k_s = 2\pi/2\Delta x = \pi/\Delta x, \]
and
\[ \sigma^*_s = \sigma \frac{\sin(k_s\Delta x)}{k_s\Delta x} = \sigma \frac{\sin \pi}{\pi} = 0. \]

No matter what the true phase speed, the shortest waves will have a numerical phase speed of 0. \( \Rightarrow \) This implies large error are present for the shortest resolvable waves!

What about longer waves? Say \( \lambda = 4\Delta x \),
\[ \lambda = 4\Delta x \Rightarrow k = \frac{\pi}{2\Delta x} \]
\[ \Rightarrow \sigma^* = \sigma \frac{\sin(\pi/2)}{\pi/2} = 0.6366\sigma. \]
Figure 1: Representation of the minimum resolvable wavelength ($\lambda$, the distance from peak to peak) on a numerical grid.

Or say $\lambda = 8\Delta x$,

$$\lambda = 8\Delta x \Rightarrow k = \frac{\pi}{4\Delta x}$$

$$\Rightarrow \sigma^* = \sigma \sin\left(\frac{\pi}{4}\right)$$

This means the longer the wave, the smaller the error in propagation speed.

3 More on Phase Speed

General Case

Let

$$\lambda = n\Delta x$$

$$\sigma^* = \sigma \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}$$

as $n \to \infty$

$$\sigma^*_{n \to \infty} = \lim_{n \to \infty} \sigma \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}.$$

Use L'Hôpital’s rule:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}.$$
So
\[ \sigma^*_{n \to \infty} = \sigma \frac{-2\pi}{n^2} \cos\left(\frac{2\pi}{n}\right) \mid_{n=\infty} \]
\[ \Rightarrow \sigma^*_{n \to \infty} = \sigma. \]

So the larger \( n \) is (ie. the wavelength) the smaller the error!

**Group Velocity**

In a wave packet (an overlay of many different waves with varying wave numbers) energy is transported with the group velocity

\[ \sigma_g = \frac{\partial \omega}{\partial k} \]

where \( \omega = \) frequency.

The analytical solution is
\[ u = u_0 e^{ik(x-\sigma t)} = u_0 e^{i(kx-\omega t)} \]

where
\[ \omega = k\sigma. \]
\[ \Rightarrow \sigma_g = \frac{\partial}{\partial k} (k\sigma) = \sigma. \]

This tells us that the group velocity equals the phase speed, so the wave is non-dispersive!

**Numerical Solution**

Let's apply group velocity to the numerical solution:
\[ \omega = k\sigma^* = k\sigma \frac{\sin(k\Delta x)}{k\Delta x} = \sigma \frac{\sin(k\Delta x)}{\Delta x} \]

so
\[ \sigma_g^* = \frac{\partial \omega}{\partial k} = \sigma \cos(k\Delta x) \]
Example: $\lambda = 2\Delta x \Rightarrow k = \pi/\Delta x$

$\Rightarrow \sigma^* = \sigma \cos \pi = -\sigma.$

Therefore, the numerical group velocity is not equal to the phase velocity as in the analytical solution. In fact it has the opposite sign!

4 Discretization Error of Backwards Differencing

We now analyse the backward in space differencing scheme. We start by writing the advection equation from (1) as

$$\frac{\partial u}{\partial t} = -\sigma \frac{\partial u}{\partial x} \Rightarrow \frac{\partial u_j}{\partial t} = -\sigma \frac{u_j - u_{j-1}}{\Delta x}.$$  

Let $u_j = U(t)e^{ijk\Delta x}$

$$\Rightarrow \frac{dU(t)}{dt} e^{ijk\Delta x} = -\sigma \left( \frac{U(t)e^{ijk\Delta x} - U(t)e^{i(j-1)k\Delta x}}{\Delta x} \right)$$

$$\Rightarrow \frac{1}{U(t)} \frac{dU(t)}{dt} = -\sigma \left( \frac{1 - e^{-ik\Delta x}}{\Delta x} \right)$$

$$= -\sigma \left( \frac{1 - (\cos k\Delta x - i\sin k\Delta x)}{\Delta x} \right)$$

$$= -ik\sigma \left( \frac{1 - (\cos k\Delta x - i\sin k\Delta x)}{ik\Delta x} \right)$$

$$= -ik\sigma^*$$

where

$$\sigma^* = \sigma \left( \frac{1 - \cos k\Delta x + i\sin k\Delta x}{ik\Delta x} \right).$$

Therefore

$$U(t) = u_0 e^{-ik\sigma^* t}$$

and

$$u_j = U(t)e^{ijk\Delta x} = u_0 e^{ik(j\Delta x - \sigma^* t)}$$
Analytic Solution: $u = u_0 e^{ik(x-\sigma t)}$

at $x = j\Delta x \rightarrow u = u_0 e^{ik(j\Delta x-\sigma t)}$.

$\Rightarrow$ Discretization error since $\sigma \neq \sigma^*$. Investigate $\sigma^*$ a bit more:

$$u_j = u_0 e^{ik[j\Delta x-\sigma t(\frac{1-cos k\Delta x}{ik\Delta x})-\sigma t(\frac{1 \sin k\Delta x}{ik\Delta x})]}$$

$$= u_0 e^{-\sigma t(\frac{1-cos k\Delta x}{\Delta x})} e^{ik[j\Delta x-\sigma t(\frac{1 \sin k\Delta x}{k\Delta x})]}.$$ 

Term containing $\sin k\Delta x/k\Delta x \rightarrow$ numerical dispersion.

Term containing $(1 - \cos k\Delta x)/\Delta x \rightarrow$ damping of solution for $\sigma > 0$, or growth and hence instability for $\sigma < 0$. 
