Here, red fluid is dense (salty), clear fluid is fresh.

“Hydraulic jump.” Sudden return to subcritical flow

Supercritical downslope flow. (Almost always unstable to shear instability.)

Here, velocity matches long wave speed (critical)

Slow incoming flow (subcritical)

Photo & video thanks to Tjipto Prastowo, ANU.
Shear Instability in the Ocean
Consider a stable density field, with parallel shear flow. When, why and how will this go unstable?
A linear stability analysis can tell us the likely modes and characteristics of instability:

- Divide flow into background and perturbation components

  \[ u(x, z, t) = \bar{u}(z) + u'(x, z, t) \]
  \[ w(x, z, t) = w'(x, z, t) \]
  \[ p(x, z, t) = \bar{p}(z) + p'(x, z, t) \]
  \[ \rho(x, z, t) = \bar{\rho}(z) + \rho'(x, z, t) \]

- Note that \( \bar{u} \), \( \bar{p} \) and \( \bar{\rho} \) satisfy the equations of motion

- Assume that perturbations \( u' \), etc., are small so cancel quadratic terms.

- Assume a wave-like form as your solution,

  \[ \psi = \hat{\psi}(z)e^{ik(x-ct)} \]

  and look for imaginary values of \( c \) (exponentially growing modes).
Linear Stability Analysis

Start with the inviscid, non-diffusive, 2D, non-rotating equations of motion:

\[
\begin{align*}
    u_t + uu_x + wu_z &= \frac{-p_x}{\rho_0} \\
    w_t + uw_x + ww_z &= \frac{-\rho g}{\rho_0} - \frac{p_z}{\rho_0} \\
    \rho_t + u\rho_x + w\rho_z &= 0 \\
    u_x + w_z &= 0
\end{align*}
\]

Manipulate to get

\[
\begin{align*}
    u'_t + uu'_x + w'u_z &= \frac{-p'_x}{\rho_0} \\
    w'_t + uu'_x &= \frac{-\rho'_g}{\rho_0} - \frac{p'_z}{\rho_0} \\
    \rho'_t + u\rho'_x + w'\rho_z &= 0 \\
    u'_x + w'_z &= 0
\end{align*}
\]
Linear Stability Analysis

\[ u_t' + \bar{u}u_x' + w'\bar{u}_z = \frac{-p_x'}{\rho_0} \]
\[ w_t' + \bar{u}w_x' = -\frac{\rho' g}{\rho_0} - \frac{p_z'}{\rho_0} \]
\[ \rho_t' + \bar{u}\rho_x' + w'\bar{\rho}_z = 0 \]
\[ u_x' + w_z' = 0 \]

Define streamfunction to satisfy continuity:

\[ u' = -\psi_z \quad w' = \psi_x \]

\[-\psi_{zt} - \bar{u}\psi_{zx} + \psi_x\bar{u}_z = \frac{-p_x'}{\rho_0} \]
\[ \psi_{xt} + \bar{u}\psi_{xx} = -\frac{\rho' g}{\rho_0} - \frac{p_z'}{\rho_0} \]

Cross-differentiate momentum equations:

\[ \nabla^2 \psi_t + \bar{u}\nabla^2 \psi_x - \bar{u}_{zz}\psi_x = \frac{-\rho_x' g}{\rho_0} \]

Substitute

\[ \psi = \psi(\hat{z})e^{ik(x-ct)} \]
\[ \rho' = \rho(\hat{z})e^{ik(x-ct)} \]

\[ \hat{\rho}(\bar{u} - c) + \hat{\psi}\bar{\rho}_z = 0 \]
\[ (\bar{u} - c)\hat{\psi}_{zz} - k^2(\bar{u} - c)\hat{\psi} - \bar{u}_{zz}\hat{\psi} = \frac{-\hat{\rho}g}{\rho_0} \]
\[ \hat{\psi}_{zz} - \left( k^2 + \frac{\bar{u}_{zz}}{(\bar{u} - c)} - \frac{N^2}{(\bar{u} - c)^2} \right) \hat{\psi} = 0 \]
Linear Stability Analysis

The Taylor-Goldstein Equation

\[
\hat{\psi}_{zz} - \left( k^2 + \frac{\bar{u}_{zz}}{(\bar{u} - c)} - \frac{N^2}{(\bar{u} - c)^2} \right) \hat{\psi} = 0
\]

How do we get from here, to this:
We could do it this way . . . but we won’t

Substitute $\hat{\psi} = \sqrt{(\bar{u} - c)} \phi$, and for simplicity here assume $\frac{d^2 \bar{u}}{dz^2} = 0$:

$$
\frac{d}{dz} \left[ (\bar{u} - c) \frac{d\phi}{dz} \right] - \left[ l^2 (\bar{u} - c) + \frac{1}{\bar{u} - c} \left( \frac{1}{4} \bar{u}_z^2 - N^2 \right) \right] \phi = 0
$$

Multiply by complex conjugate $\phi^*$ and integrate

$$
\int_0^H (N^2 - \frac{1}{4} \bar{u}_z^2) \frac{|\phi|^2}{\bar{u} - c} \, dz = \int_0^H (\bar{u} - c) \left( |\frac{d\phi}{dz}|^2 + l^2 |\phi|^2 \right) \, dz
$$

$$
c_i \left[ \int_0^H (N^2 - \frac{1}{4} \bar{u}_z^2) \frac{|\phi|^2}{|\bar{u} - c|^2} \, dz + \int_0^H \left( |\frac{d\phi}{dz}|^2 + l^2 |\phi|^2 \right) \, dz \right] = 0
$$

Right hand side is $> 0$. Hence $c_i = 0$ if $LHS > 0$ (so $c = c_r \pm ic_i$ is real). Hence flow is stable for $N^2 - \frac{1}{4} \bar{u}_z^2 > 0$.

**Richardson number** $Ri = \frac{N^2}{(d\bar{u}/dz)^2} > 1/4$: sufficient for stability

**Kelvin-Hemholtz instability**

$Ri < 1/4$ is a necessary condition for K-H instability (though not sufficient, this is also found to be a reliable indicator of instability)
Kelvin-Helmholtz Instability

The Taylor-Goldstein Equation

\[ \hat{\psi}_{zz} - \left( k^2 + \frac{\bar{u}_{zz}}{(\bar{u} - c)} - \frac{N^2}{(\bar{u} - c)^2} \right) \hat{\psi} = 0 \]

Solution for this case is

\[ \hat{\psi} = e^{-k|z-d|-i\theta} - e^{-k|z+d|+i\theta} \]

and eigenvalues are

\[ \left( \frac{c}{U_0} \right)^2 = \frac{(1 - 2kd)^2 - e^{-4kd}}{(2kd)^2} \]

Instability occurs when \( kd < 0.64 \).

Baines & Mitsudera (1994)
Kelvin-Helmholtz Instability

- Vorticity wave on each interface
- Phase locking
- Mutual amplification

Baines & Mitsudera (1994)
Shear Instability: When waves of opposite celerity coalesce.
Addition of stratification stabilises the flow.

Define Bulk Richardson Number:

\[ J \equiv \frac{g \Delta \rho d}{2 \rho_0 U_0^2} = \frac{g' d}{2 U_0^2} \]
Kelvin-Helmholtz Instability

Growth rate as a function of $k$ and $J$:
Holmboe’s Instability

[Image of a green and black gradient]
Holmboe’s Instability

\[ \rho(z) \]

\[ \rho_0 \]

\[ \rho_0 + \Delta \rho \]

\[ U(z) \]

\[ U_0 \]

\[ z = 0 \]

\[ z = d \]

\[ z = Rd \]

\[ z = -d \]
Holmboe’s Instability

- Now have two vorticity and two internal gravity waves
- Low $J$: vorticity waves phase lock $\rightarrow$ KH
- Higher $J$: Each vorticity wave phase locks with a gravity wave!
- No value of $J$ for which flow is stable.

Baines & Mitsudera (1994)
Holmboe’s Instability

Inviscid stability diagram

Bulk Richardson Number, $J$

Wavenumber

Haigh & Lawrence (1998)
Summary

- Shear destabilises
- Stratification stabilises
- $Ri$ characterises these two competing effects
- Instability results from phase locking waves of opposing celerity
- Type of instability is sensitive to the vertical profile of density and shear.